Symmetry breaking in non-commutative cut-off field theories

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Abstract. The relation between symmetry breaking in non-commutative cut-off field theories and transitions to inhomogeneous phases in condensed matter and in finite density QCD is discussed. The non-commutative dynamics, with its peculiar infrared-ultraviolet mixing, can be regarded as an effective description of the mechanisms which lead to inhomogeneous phase transitions and a roton-like excitation spectrum.

The relation between symmetry breaking in quantum field theory and phase transitions in condensed matter is well known. The previous correspondence is usually restricted to the case of constant order parameter because this guarantees the translational and rotational invariance of the field theory. On the other hand, in condensed matter and, more recently, in high density QCD one considers transitions from homogeneous to inhomogeneous phases with non-constant order parameters. For a bosonic system, according to [1], this kind of transition is associated with an order parameter which for large distances is an oscillating function and with a roton-like behavior of the excitation spectrum. For the fermionic systems it is possible to build inhomogeneous superconducting states, with energy lower than the BCS state, where the particle-particle (p-p) Cooper pairs or the particle-hole (p-h) pairs have non-zero total momentum and then the corresponding fermionic condensates are not uniform. The former phase is described in condensed matter systems by the LOFF state [2] and in QCD is related with a diquark condensation with interesting theoretical and phenomenological implications [3, 4]. The latter has been originally proposed in condensed matter by Overhauser [5] (the (p-h) instability is called spin density wave) and in QCD the corresponding phase, called chiral density wave, can compete with the QCD-BCS phase in the strong coupling regime and in an intermediate region of the chemical potential [6].

In [7, 8, 9, 10, 11] it has been shown that the previous condensed matter and QCD inhomogeneous transitions are also typical features of quantum field theory with non-commutative coordinates, $[x_\mu, x_\nu] = i\theta_{\mu\nu}$. As discussed in this talk, in the non-commutative generalization of $\lambda\phi^4$ theory [7, 8, 11] the spontaneous symmetry breaking occurs for a non-uniform stripe phase and in the non-commutative Gross-Neveu (GN) model there is an inhomogeneous chiral symmetry breaking corresponding to spin density waves [9]. These results are mainly due to the infrared/ultraviolet (IR/UV) connection which characterizes the non commutative field theories (for a review see [12]).

We shall first discuss the relation between the symmetry breaking in non-commutative self-interacting scalar field theory and the roton excitation in BEC and then we consider

the fermionic systems. On general grounds, in a bosonic condensate the roton spectrum is due to a non-local interatomic potential $V(\vec{r} - \vec{r}')$, with a momentum dependent Fourier transform. Since the BEC with the local (pseudo)potential $\delta(\vec{r} - \vec{r}')$ is analogous to the spontaneous symmetry breaking in $\lambda\phi^4$ theory, one can assume that some relevant physical effects due to the non-local repulsive interaction can be described by generalizing the self-interacting field theory in such a way to introduce an effective non-local coupling.

A simple approach is to consider the non-commutative $\lambda\phi^4$ theory with action

$$S(\phi) = \int d^4x \left(\frac{1}{2}\partial_\mu\phi \ \partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^{4*} \right) \tag{1}$$

where the star (Moyal)product is defined by $(i, j = 1, ., 4)$

$$\phi^{4*}(x) = \phi(x) * \phi(x) * \phi(x) * \phi(x) =$$
$$\exp\left[\frac{i}{2} \sum_{ij} \theta_{\mu\nu}\partial^\mu_{x_i}\partial^\nu_{x_j} \right] \left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \right)\Big|_{x_i=x} \tag{2}$$

The "deformation" of the self-interaction term by the Moyal product gives a momentum dependent repulsive effect which is responsible, as we shall see below, for the roton spectrum and for the phase transition to an inhomogeneous background. In [8], for $\theta_{ij} = \varepsilon_{ijk}\theta^k$ with $\vec{\theta} = (0, 0, \theta)$, the spontaneous symmetry breaking for the theory in Eqs. (1) and (2) has been analyzed with the following results: **1)** the transition occurs to a stripe phase where the order parameter is $\phi(\vec{x}) = A\cos\vec{Q}\cdot\vec{x}$; **2)** A, Q and the energy excitation $\omega(p)$ are fixed by minimizing the energy; **3)** $\vec{Q}$ is orthogonal to $\vec{\theta}$ and Q is small for large θ; **4)** the excitation spectrum can be approximated by $\omega^2(\vec{p}) = p^2 + M^2(\vec{p})$ where the function $M(\vec{p})$ will be discussed later. As discussed in detail in [8], since Q is small, the inhomogeneous background is a smooth function of x and then, the breaking of translational (smooth) and rotational invariance is approximated by a translational invariant propagator with a momentum dependent mass term. In the particular case $\vec{Q} = (Q/\sqrt{2}, Q/\sqrt{2}, 0)$ and large values of $\theta\Lambda^2$ (Λ is the UV cut-off), it turns out that $Q^2/\Lambda^2 = (\lambda/24\pi^2)^{1/2}(1/\theta\Lambda^2)$ and that $M(\vec{p})$, for small p, is given by

$$M^2(\vec{p})\big|_{p\to 0} \sim \alpha + \frac{\lambda}{6\pi^2}\frac{1}{|\vec{p}\times\vec{\theta}|^2} \tag{3}$$

where α is a constant and $\times$ indicates the usual vector product. The peculiar behavior for small p of the last term in the previous equation is due to the IR/UV connection of the non-commutative field theory and gives a divergent mass term in the IR region and a minimum in the irreducible two-point function. However the effective theory has a natural self-generated IR cut-off Q where it is more correct to cut the small momenta and then the excitation spectrum should be correctly identified for $p \geq Q$ by the previous expressions for $\omega(p)$ and $M^2(p)$. It has a roton-like dip at a typical scale of order Q.

The Moyal-deformed term in Eq. (1) can mimic effective interactions which are non-local and globally repulsive. Then, the previous results of the non-commutative

theory can describe some interesting physical effects of the Bose -Einstein trapped condensates where the non-contact repulsive interaction is dominant. This analysis has been performed in detail in ref. [10] where the correlation between roton-like spectrum and non-uniform background of the non commutative theory has been compared with the results obtained for BE trapped condensates, in analogous dynamical conditions.

Let us now consider the informations coming from non-commutative effective field theories for fermionic systems. In [9] the transition from homogeneous to inhomogeneous phase has been obtained by generalizing the GN model, in four dimensions, to the non-commutative case with lagrangian

$$L(x) = i\bar{\psi}_\alpha \slashed{\partial} \psi_\alpha + g\bar{\psi}_\alpha * \psi_\alpha * \bar{\psi}_\beta * \psi_\beta - g\bar{\psi}_\alpha * \bar{\psi}_\beta * \psi_\alpha * \psi_\beta \ . \tag{4}$$

For g larger than some critical value, one finds, as in tha commutative case, chiral symmetry breaking but, this time, in an inhomogeneous phase where the pair correlation function has a dependence on a total momentum, $\vec{P}$ of the ("Cooper") pair, with $P/\Lambda \simeq (1/\theta\Lambda^2)$.The order parameter turns out to be an oscillating function of $\vec{x}$ and one has the breaking of translational, rotational and chiral invariance : $< \bar{\psi}(x)\psi(x) >= [1 + cP^2\cos(Px)] < \bar{\psi}\psi >_0$, where $< \bar{\psi}\psi >_0$ is the constant order parameter of the commutative case and c is a numerical constant. Also in this case [9] the spectrum has roton-like dip in the plane orthogonal to $\vec{\theta}$ and one recovers the dynamical relation with the non-uniform ground state [1]. The previous field theoretical model is then analogous to a system with a non-local, strong four-fermion interaction, with an inhomogeneous phase where the particle-hole (p-h) pairs have non-zero total momentum.As discussed, this phase in QCD has an energy close to the BCS phase only for strong coupling , corresponding to intermediate value of the density, that is before entering the perturbative regime where the QCD- LOFF phase is realized.

Finally, the non commutative field theoretical results indicate that the phase transitions to inhomogeneous condensates are first order and one expects similar behavior for the corresponding condensed matter and QCD systems.

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